BIBLIOGRAPHY

1. Bolotin, V. V., The edge effect in the oscillations of elastic shells. PMM Vol. 24, №5, 1960.
2. Kornev, V. M., Stability of a circular cylindrical shell, loaded by external transverse pressure, taking edge effect into account. Inzh. Zh., Mekh. Tverdogo Tela, №3, 1967.
3. Vishik, M. I. and Liusternik, L. A., Solution of some problems on perturbations in the case of matrices and self-adjoint and non-self-adjoint differential equations. I. Usp. Matem. Nauk, Vol. 15, №3, 1960.
4. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. 2. Moscow-Leningrad, Gostekhizdat, 1951.
5. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products. Moscow, Fizmatgiz, 1963.
6. Vlasov, V. Z., Selected Works, Vol. 1. Moscow, Akad. Nauk SSSR Press, 1962.
7. Bolotin, V. V., On the density of the distribution of natural frequencies of thin elastic shells. PMM Vol. 27, №2, 1963.
8. Bolotin, V. V., Eigenvalue density in problems of elastic plate and shell vibrations. Trans. of VI All-Union Conf. on Plates and Shells Theory (Baku, 1966). Moscow, "Nauka", 1966.
9. Donnell, L. H., A new theory for the buckling of thin cylinders under axial compression and bending. Trans. ASME, Vol. 56, №10, 1934.
10. Kármán, Th. and Tsien, H. S., The buckling of thin cylindrical shells under axial compression. J. Aero. Sci., Vol. 8, №8, 1941.
11. Vol'mir, A. S., Stability of Elastic Systems. Moscow, Fizmatgiz, 1963.

Translated by M. D. F.

# ON EXTREMAL STRESSES IN THE PLANE PROBLEM OF THE THEORY OF ELASTICITY

PMM Vol. 35, №2, 1971, pp. 369-375

S. A. KAS'IANIUK and T. I. TKACHUK
(Kiev)

(Received July 21, 1969)

The problem of extremal stresses in the first fundamental problem for a half-plane and a circle depending on the stress distribution on the contour is studied by using estimates for the integral operators of plane elasticity theory. S. A. Kas'ianiuk solved the problems for the half-plane and G. I. Tkachuk for the circle.

It is known from [1], p. 293 and from [2] that the stress components $X_x$, $X_y$, $Y_y$ at the point $z = x + iy$ in the first fundamental plane problem for the half-plane $y < 0$ are defined in terms of the normal $N(t)$ and tangential $T(t)$ stresses given along the $x$-axis by using the equalities

$$X_x + Y_y = 4\operatorname{Re}\Phi(z) \tag{0.1}$$

$$Y_y - X_x + 2iX_y = 2\left[\bar{z}\Phi'(z) + \Psi(z)\right]$$

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{N(t) - iT(t)}{(z-t)} dt \qquad \text{(cont.)}$$

$$\Psi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{N(t) + iT(t)}{(z-t)} dt + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{N(t) - iT(t)}{(z-t)^2} dt$$

In the problem for the circle $|z| \leqslant R$, as is known from [2], the tangential and normal stresses $R_\theta$ and $R_r$ at the point $z = re^{i\theta}$, $r \neq 0$, are defined (in a polar coordinate system) by the equalities

$$2R_\theta r^2 = \frac{R^2 - r^2}{2} \frac{\partial \Omega}{\partial \theta} + \frac{R^2}{\pi} \int_0^{2\pi} g_1(t) T(t) dt \tag{0.2}$$

$$2R_r r^2 = r \frac{R^2 - r^2}{2} \frac{\partial \Omega}{\partial r} + \frac{R^2 - r^2}{\pi} \int_0^{2\pi} g_2(t) T(t) dt + \frac{r^2}{\pi} \int_0^{2\pi} g_1(t) N(t) dt$$

$$\Omega = \frac{1}{\pi} \int_0^{2\pi} [g_1(t) N(t) - g_2(t) T(t)] dt$$

Here and henceforth, we have introduced the notation

$$g_1(t) = \frac{R^2 - r^2}{g(t)}, \quad g_2(t) = \frac{2rR \sin(\theta - t)}{g(t)}, \quad g(t) = R^2 - 2rR\cos(\theta - t) + r^2 \tag{0.3}$$

and $N(t)$ and $T(t)$ are the corresponding, given normal and tangential stresses on the circle $z = Re^{it}$. Therefore, the stress components at an arbitrary point $z$ of the domain under consideration are functionals defined on sets of functions $N(t)$ and $T(t)$. If it be required that the functions $N(t)$ and $T(t)$ satisfy some constraints, for example; either

$$|N(t)| \leqslant l, \qquad T(t) \equiv 0 \tag{0.4}$$

or

$$|T(t)| \leqslant l, \qquad N(t) \equiv 0$$

or

$$\int_\Gamma [N^2(t) + T^2(t)] dt \leqslant l^2$$

etc., then the problem of seeking the extremal values of the quantities

$$X_y, Y_y, R_\theta, R_r, \quad \sqrt{X_y^2 + Y_y^2}, \quad \sqrt{R_\theta^2 + R_r^2}$$

etc., and the functions $N(t)$ and $T(t)$ realizing them, can be posed.

To solve the formulated problems herein, various modifications are utilized of the Cauchy-Buniakowski inequality in Banach space [3]

$$|X(x)| \leqslant \|X^{**}\| \cdot \|x\| \tag{0.5}$$

where $X(x)$ is a linear functional, $\|x\|$ is the norm of the element and $\|X^{**}\|$ is the norm of the functional.

**1.** Let us first examine the case when $T(t) \equiv 0$ along the half-plane boundary, and $N(t)$ satisfies the conditions

$$N(t) = 0 \quad (|t| > a), \qquad |N(t)| \leqslant l \qquad (|t| \leqslant a) \tag{1.1}$$

caused by physical considerations [1]. Then

$$X_y = \frac{2}{\pi} \int_{-a}^{a} \frac{h_1(t)}{h^2(t)} N(t)\,dt, \quad Y_y = -\frac{2}{\pi} \int_{-a}^{a} \frac{y^3}{h^2(t)} N(t)\,dt \tag{1.2}$$

Here as well as henceforth, the following notation is introduced

$$h_1(t) = y^2(t - x), \quad h_2(t) = y(t - x)^2, \quad h(t) = (x - t)^2 + y^2 \tag{1.3}$$

Utilizing an estimate of the form (0. 5) for the integrals (1. 2) with the constraints (1.1), we rapidly obtain that the maximum of $X_y$ in $N(t)$ is realized only by a function $N_0(t)$ of the form:

1) if $|x| \leqslant a$

$$N_0(t) = 0 \ (|t| > a), \ N_0(t) = -l \ (-a \leqslant t < x), \ N_0(t) = l \ (x \leqslant t \leqslant a)$$

$$X_y = \frac{l}{\pi}[2 - \omega(x, y) - \omega(-x, y)]$$

$$Y_y = \frac{l}{\pi}[\varepsilon(x, y) - \varepsilon(-x, y)] + \frac{ly}{\pi}[\theta(x, y) - \theta(-x, y)]$$

$$\theta(x, y) = \frac{(a + x)}{h(-a)}, \quad \omega(x, y) = \frac{y^2}{h(-a)}, \quad \varepsilon(x, y) = \operatorname{arc\,tg} \frac{(a + x)}{y}$$

2) if $x < -a$ (or $x > a$)

$$N_0(t) = 0 \quad (|t| > a), \quad N_0(t) = \pm l \quad (|t| \leqslant a)$$

$$X_y = \mp \frac{l}{\pi}[\omega(x, y) - \omega(-x, y)]$$

$$Y_y = \pm \frac{ly}{\pi}[\theta(x, y) + \theta(-x, y)] \pm \frac{l}{\pi}[\varepsilon(x, y) + \varepsilon(-x, y)]$$

(the lower signs correspond to the case $x > a$). From the same considerations it is clear that the maximum of $Y_y$ in $N(t)$ is realized only by a function $N_0(t)$ of the form

$$N_0(t) = l \quad (|t| \leqslant a), \quad N_0(t) = 0 \quad (|t| > a)$$

$$Y_y = -\frac{ly}{\pi}[\theta(x, y) + \theta(-x, y)] - \frac{l}{\pi}[\varepsilon(x, y) + \varepsilon(-x, y)]$$

$$X_y = \frac{l}{\pi}[\omega(x, y) - \omega(-x, y)]$$

Now let us examine the case when $N(t) \equiv 0$ along the half-plane boundary and $T(t)$ satisfies the conditions

$$T(t) = 0 \quad (|t| > a), \qquad |T(t)| \leqslant l \ (|t| \leqslant a) \tag{1.4}$$

Then

$$X_y = -\frac{2}{\pi} \int_{-a}^{a} \frac{h_2(t)}{h^2(t)} T(t)\,dt, \quad Y_y = \frac{2}{\pi} \int_{-a}^{a} \frac{h_1(t)}{h^2(t)} T(t)\,dt \tag{1.5}$$

Utilizing an estimate of the form (0. 5) for the integrals (1. 5) with the constraints (1.4), we obtain that the maximum of $X_y$ in $T(t)$ is realized only by a function $T_0(t)$ of the form

$$T_0(t) = l \ (|t| \leqslant a), \ T_0(t) = 0 \ (|t| > a)$$

$$X_y = \frac{ly}{\pi}[\theta(x, y) + \theta(-x, y)] - \frac{l}{\pi}[\varepsilon(x, y) + \varepsilon(-x, y)]$$

$$Y_y = \frac{l}{\pi}[\omega(x, y) - \omega(-x, y)]$$

the maximum of $Y_y$ in $T(t)$ is realized only by a function $T_0(t)$ of the form:

1) if $|x| \leqslant a$

$$T_0(t) = 0 \quad (|t| > a), \quad T_0(t) = -l \; (-a \leqslant t < x), \quad T_0(t) = l \quad (x \leqslant t \leqslant a)$$

$$Y_y = \frac{l}{\pi}[2 - \omega(x, y) - \omega(-x, y)]$$

$$X_y = \frac{ly}{\pi}[\theta(x, y) - \theta(-x, y)] + \frac{l}{\pi}[\varepsilon(x, y) - \varepsilon(-x, y)]$$

2) if $x < -a$ (or $x > a$)

$$T_0(t) = 0 \quad (|t| > a), \quad T_0(t) = \pm l \quad (|t| \leqslant a)$$

$$Y_y = \mp \frac{l}{\pi}[\omega(x, y) - \omega(-x, y)]$$

$$X_y = \pm \frac{ly}{\pi}[\theta(x, y) + \theta(-x, y)] \mp \frac{l}{\pi}[\varepsilon(x, y) + \varepsilon(-x, y)]$$

(the lower signs correspond to the case $x > a$).

Finally, let us examine the case when the stresses along the half-plane contour satisfy the condition

$$\int_{-\infty}^{\infty} [N^2(t) + T^2(t)]\, dt \leqslant l^2 \tag{1.6}$$

Taking account of the notation (1.3), we have from the equalities (0.1)

$$X_y = \frac{2}{\pi}\int_{-\infty}^{\infty} [h_1(t) N(t) - h_2(t) T(t)] \frac{dt}{h^2(t)} \tag{1.7}$$

$$Y_y = \frac{2}{\pi}\int_{-\infty}^{\infty} [-y^3 N(t) + h_1(t) T(t)] \frac{dt}{h^2(t)}$$

and in this case the inequality (0.5) becomes

$$\begin{vmatrix} l^2 A_{11} - X_y^2 & l^2 A_{12} - X_y Y_y \\ l^2 A_{21} - X_y Y_y & l^2 A_{22} - Y_y^2 \end{vmatrix} \geqslant 0 \tag{1.8}$$

$$A_{11} = \frac{4}{\pi^2}\int_{-\infty}^{\infty} \frac{y h_2(t)}{h^3(t)}\, dt = -\frac{1}{2\pi y}$$

$$A_{22} = \frac{4}{\pi^2}\int_{-\infty}^{\infty} \frac{y^4}{h^3(t)}\, dt = -\frac{3}{2\pi y}$$

$$A_{12} = A_{21} = -\frac{4}{\pi^2}\int_{-\infty}^{\infty} \frac{y h_1(t)}{h^3(t)}\, dt = 0$$

As is known from [3], the equality sign is realized in (1.8) only by those $X_y$ and $Y_y$ which correspond to the functions

$$N_0(t) = \frac{2}{\pi}\frac{[\xi_1 h_1(t) - \xi_2 y^3]}{h^2(t)}, \quad T_0(t) = \frac{2}{\pi}\left[\frac{\xi_1 h_2(t) + \xi_2 h_1(t)}{h^2(t)}\right] \tag{1.9}$$

with appropriate $\xi_1$ and $\xi_2$.

Consequently

$$-\frac{2\pi y}{l^2} X_y^2 - \frac{2}{3}\frac{\pi y}{l^2} Y_y^2 \leqslant 1$$

$$\max_{N(t),T(t)} X_y = \frac{l}{\sqrt{-2\pi y}}, \quad \max_{N(t),T(t)} Y_y = \frac{l\sqrt{3}}{\sqrt{-2\pi y}} \quad \text{(cont.)}$$

$$\max_{N(t),T(t)} |X_y + iY_y| = \frac{l\sqrt{3}}{\sqrt{-2\pi y}}.$$

The functions $N_0(t)$ and $T_0(t)$ realizing $\max_{N(t),T(t)} X_y$ have the form

$$N_0(t) = \frac{2}{\pi}\frac{l}{\sqrt{-2\pi y}}\frac{h_1(t)}{h^2(t)}, \qquad T_0(t) = -\frac{2}{\pi}\frac{l}{\sqrt{-2\pi y}}\frac{h_2(t)}{h^2(t)} \tag{1.10}$$

while those realizing $\max_{N(t),T(t)} Y_y$ and $\max_{N(t),T(t)} |X_y + iY_y|$ have the form

$$N_0(t) = -\frac{2}{\pi}\frac{l\sqrt{3}}{\sqrt{-2\pi y}}\frac{y^3}{h^2(t)}, \quad T_0(t) = \frac{2}{\pi}\frac{l\sqrt{3}}{\sqrt{-2\pi y}}\frac{h_1(t)}{h^2(t)} \tag{1.11}$$

The values of $\xi_1$ and $\xi_2$ for the extremal functions (1.10) and (1.11) are found from the equations

$$\frac{l}{\sqrt{-2\pi y}} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{h_1(t)N_0(t) - h_2(t)T_0(t)}{h^2(t)}\,dt$$

$$0 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-y^3 N_0(t) + h_2(t)T_0(t)}{h^2(t)}\,dt$$

$$0 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{h_1(t)N_0(t) - h_2(t)T_0(t)}{h^2(t)}\,dt$$

$$\frac{l\sqrt{3}}{\sqrt{-2\pi y}} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-y^3 N_0(t) + h_2(t)T_0(t)}{h^2(t)}\,dt$$

Here $N_0(t)$ and $T_0(t)$ are determined by the equalities (1.9).

**2.** The first fundamental plane problem for a circle is investigated in perfect analog. Let us first examine the problem of maximal $R_\theta$ and $R_r$ at the point $z = re^{i\theta}$, $r \neq 0$, of the circle $|z| \leqslant R$, when only a normal stress

$$N(t), \qquad |N(t)| \leqslant l \tag{2.1}$$

is applied to points of the boundary circle $z = Re^{it}$, $0 \leqslant t \leqslant 2\pi$. In this case $T(t) \equiv 0$, hence we obtain from (0.2)

$$R_\theta = \frac{R^2 - r^2}{4\pi r^2}\int_0^{2\pi} \frac{g_2(t)}{g(t)} N(t)\,dt \tag{2.2}$$

$$R_r = \frac{(R^2 - r^2)^2}{2\pi r}\int_0^{2\pi} \frac{R\cos(t-\theta) - r}{g^2(t)} N(t)\,dt$$

The maximum of $R_\theta$ in $N(t)$ is realized, on the basis of (0.5), only by a function $N_0(t)$ of the form

$$N_0(t) = l \quad (\theta \leqslant t \leqslant \theta + \pi), \quad N_0(t) = -l \; (0 \leqslant t < \theta, \; \pi + \theta < t \leqslant 2\pi)$$

moreover

$$R_\theta = \frac{2Rl}{\pi r}, \qquad R_r = 0$$

The maximum of $R_r$ in $N(t)$ is realized only by a function $N_0(t)$ of the form

$$N_0(t) = -l \quad (\vartheta \leqslant t \leqslant 2\pi - \vartheta), \quad N_0(t) = l \quad (0 \leqslant t < \vartheta,\ 2\pi - \vartheta < t \leqslant 2\pi)$$

$$\vartheta = \theta + \arccos \frac{r}{R}$$

moreover

$$R_r = \frac{2l}{\pi r} \sqrt{R^2 - r^2} + \frac{4l}{\pi} \operatorname{arc\,tg} \sqrt{\frac{R+r}{R-r}}, \qquad R_\theta = 0$$

Now let us examine the case when

$$z = Re^{it}, \qquad 0 \leqslant t \leqslant 2\pi, \qquad N(t) \equiv 0$$

along the boundary of the circle and the tangential stresses satisfy the constraint

$$|T(t)| \leqslant l \tag{2.3}$$

In this case

$$R_\theta = \frac{R(R^2 - r^2)}{2\pi r^2} \int_0^{2\pi} \frac{R(R^2 + 3r^2) - r(3R^2 + r^2)\cos(\theta - t)}{g^2(t)} T(t)\, dt \tag{2.4}$$

$$R_r = \frac{R^2 - r^2}{4\pi r^2} \int_0^{2\pi} \frac{g_2(t)[R^2 + 3r^2 - 4Rr\cos(t - \theta)]}{g(t)} T(t)\, dt$$

Since $R(3r^2 + R^2) > r(3R^2 + r^2)$ for $r < R$, then the maximum of $R_\theta$ in $T(t)$ realized only by the function $T_0(t) \equiv l$ and moreover

$$R_\theta = \frac{lR^2}{r^2}, \quad R_r = 0 \quad (r \neq 0)$$

Two cases should be separated in seeking the maximum of $R_r$ in $T(t)$ :

1) If $0 < r \leqslant 1/3R$, then the maximum of $R_r$ in $T(t)$ is realized only by a function $T_0(t)$ of the form

$$T_0(t) = l \quad (\theta \leqslant t \leqslant \theta + \pi), \quad T_0(t) = -l \quad (0 \leqslant t < \theta,\ \theta + \pi < t \leqslant 2\pi)$$

moreover

$$R_r = \frac{2l}{\pi} \frac{R^2 - r^2}{r^2} \ln \frac{R+r}{R-r} - \frac{2lR}{\pi r}, \qquad R_\theta = 0$$

2) If ${}^1/_3 R < r < R$, then the maximum of $R_r$ in $T(t)$ is realized only by a function $T_0(t)$ of the form $T_0(t) = l \quad (\theta \leqslant t \leqslant t_1,\ \pi + \theta \leqslant t \leqslant t_2)$

$$T_0(t) = -l \quad (0 \leqslant t < \theta,\ t_1 < t < \pi + \theta,\ t_2 < t \leqslant 2\pi)$$

$$t_1 = \theta + \arccos \frac{R^2 + 3r^2}{4\pi R}, \qquad t_2 = 2\pi + 2\theta - t_1$$

moreover

$$R_r = -l \frac{R^2 - 3r^2}{\pi r^2} + 2\ln 2 \frac{l(R^2 - r^2)}{\pi r^2}, \qquad R_\theta = 0$$

Finally, when the stresses along the boundary of the circle $|z| \leqslant R$ satisfy the condition

$$\int_0^{2\pi} [N^2(t) + T^2(t)]\, dt \leqslant l^2 \tag{2.5}$$

we have

$$R_\theta = \frac{1}{4\pi R^2} \int_0^{2\pi} \{g_1(t)[2R^2(3r^2 + R^2) - 2Rr(r^2 + 3R^2)\cos(\theta - t)]\, T(t) -$$

$$- g_2(t)(R^2 - r^2)^2 N(t)\} \frac{dt}{g(t)}$$

$$R_r = \frac{R^2 - r^2}{4\pi r^2} \int_0^{2\pi} \{g_2(t)[3r^2 + R^2 - 4Rr\cos(\theta - t)]\, T(t) +$$

$$+ g_1(t)\, 2r\, [R\cos(\theta - t) - r]\, N(t)\} \frac{dt}{g(t)}$$

In this case the inequality (0.5) becomes

$$\begin{vmatrix} l^2B_{11} - R_\theta{}^2 & -R_\theta R_r \\ -R_\theta R_r & l^2B_{22} - R_r{}^2 \end{vmatrix} \geqslant 0 \qquad (2.6)$$

$$B_{11} = \frac{1}{4\pi r^2} \int_0^{2\pi} g_1^2(t)\, [2R^2(3r^2 + R^2) - 2Rr(r^2 + 3R^2)\cos(\theta - t)]^2 \frac{dt}{g^2(t)} +$$

$$+ \frac{(R^2 - r^2)^4}{4\pi r^2} \int_0^{2\pi} \frac{g_2^2(t)}{g^2(t)}\, dt$$

$$B_{22} = \frac{(R^2 - r^2)^2}{4\pi r^2} \int_0^{2\pi} \{g_2{}^2(t)\, [3r^2 + R^2 - 4Rr\cos(\theta - t)]^2 +$$

$$+ 4r^2 g_1{}^2(t)\, [R\cos(\theta - t) - r]^2\} \frac{dt}{g^2(t)}$$

Hence, the ellipse of values of $R_\theta$ and $R_r$ is defined by the inequality

$$\frac{R_\theta{}^2}{l^2B_{11}} + \frac{R_r{}^2}{l^2B_{22}} \leqslant 1$$

The boundary points of this ellipse correspond only to functions of the form

$$T_0(t) = \frac{1}{4\pi r^2} \frac{g_1(t)}{g(t)} \frac{l\cos\alpha}{\sqrt{B_{11}}} [2R^2(3r^2 + R^2) -$$

$$- 2Rr(r^2 + 3R^2)\cos(\theta - t)] + \frac{1}{4\pi r^2} \frac{g_2(t)}{g(t)} \frac{(R^2 - r^2)\, l\sin\alpha}{\sqrt{B_{22}}} [3r^2 +$$

$$+ R^2 - 4Rr\cos(\theta - t)]$$

$$N_0(t) = -\frac{1}{4\pi r^2} \frac{g_2(t)}{g(t)} \frac{l\cos\alpha}{\sqrt{B_{11}}} (R^2 - r^2)^2 +$$

$$+ \frac{1}{4\pi r^2} \frac{g_1(t)}{g(t)} \frac{l\sin\alpha}{\sqrt{B_{22}}}\, 2r(R^2 - r^2)\, [R\cos(\theta - t) - r].$$

These functions realize the maximum of $R_\theta$ for $\alpha = 0$ and the maximum of $R_r$ for $\alpha = {}^1/_2\, \pi$.

## BIBLIOGRAPHY

1. Muskhelishvili, N. I., Some Fundamental Problems of Mathematical Elasticity Theory. Moscow, "Nauka", 1966.
2. Kolosov, G. V., Application of Complex Function Diagrams of Elasticity Theory. Leningrad-Moscow, Principal Ed. of General Techn. Disciplines, 1935.
3. Krein, S. G., ed., Functional Analysis (Library of Mathematical Handbooks). Moscow, "Nauka", 1964.

Translated by M. D. F.